Chern-Simons Field Theory and Generalizations of Anyons[1,2]

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Abstract

It is well known that charges coupled to a pure Chern-Simons gauge field in (2+1) dimensions undergo an effective change of statistics, i.e., become anyons. We will consider several generalizations thereof, arising when the gauge field is more general. The first one is "multispecies anyons"—charged particles of several species coupled to one, or possibly several, Chern-Simons fields. The second one is finite-size anyons, which are charged particles coupled to a gauge field described by the Chern-Simons term *plus* some other term. In fact, rigorously speaking, quasielectrons and quasiholes in the fractional quantum Hall effect are multispecies finite-size anyons. The third one is an analog of finite-size anyons which arises in a model with a mixed Chern-Simons term; notably, this model is P,T-invariant, which opens the way for practical applications even when there is no parity-breaking magnetic field.

1 Introduction

The notion of statistics of identical particles has to do with the basic principles of quantum theory. Twenty years ago it was understood [1] (and later on confirmed in different ways [2, 3]) that this notion is by far more rich than one used to think. Namely, the common statement that there can be only two types of statistics—Bose and Fermi—is actually true only in three (or higher) dimensional space. In two dimensions (as well as in one dimension, which we will not discuss here), a continuous family of statistics is possible, characterized by a real number—*statistics parameter* α, such that $\alpha = 0$ corresponds to bosons and $\alpha = 1$ to fermions.

Speaking a bit loosely, here is the way to see it. Imagine two identical particles in a plane being interchanged in an anticlockwise manner. As a result, their wave function can only pick up a phase factor, which we will denote as $\exp[i\pi\alpha]$; without loss of generality, we can then assume $0 \leq \alpha < 2$. Repeating this operation twice means, in the frame of reference of one of the particles, that the second one is pulled all the way around it. The corresponding phase factor is $\exp[2i\pi\alpha]$. See Fig. 1.

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Fig. 1.

Now, in three dimensions, the second operation corresponds to "doing nothing", because the closed line that the second particle follows can be continuously deformed into a point. Hence the condition $\exp[2i\pi\alpha] = 1$ and the usual conclusion that α can only be 0 or 1. In two dimensions, however, no such deformation is possible, so there is no such condition and α can be *any* (real) number; hence the name *anyons* [3].

A natural question now is about the relevance of all this to "real life"—since all real particles *may* live in three dimensions and therefore can only be bosons or fermions. The answer is actually positive and is provided (maybe not uniquely) by the Chern-Simons gauge field model. If there is a conserved current j_μ and a gauge field A_μ and the Lagrangian of the theory is

$$\mathcal{L} = \mathcal{L}_G - j_\mu A^\mu \ , \tag{1}$$

$$\mathcal{L}_G = \frac{1}{2} k \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \ , \tag{2}$$

then a pointlike charge $j_\mu = e\delta_\mu^0 \delta^2(\mathbf{r})$ gives rise to a vector potential with only the angular component different from zero,

$$A_\phi(\mathbf{r}) = -\frac{\alpha}{er} \tag{3}$$

where $\alpha = e^2/2\pi k$, $r = |\mathbf{r}|$. In other words, a charge is at the same time a "magnetic flux point", and interchanging charges gives rise to a phase factor $\exp[2i\pi\alpha]$, as in the Aharonov-Bohm effect. Thus, *charge-flux composites* are anyons: the coupling between charges and fluxes "mimics" anyon statistics.

This paper consists of three short stories, each of which is in principle self-contained but which are all connected by the fact that they have to do with generalizations of the above simple picture.

2   Multispecies anyons

A further possibility specific for two dimensions is to have nontrivial statistics of *distinguishable particles* [4, 5, 6]. Imagine that particles in Fig. 1 are not identical. Their interchange can then change the wave function in an arbitrary way, but pulling one around the other one can only bring up a phase factor—because the particles come back to their original positions. The very same considerations as above imply that this factor has to be equal to unity in three dimensions but

can be arbitrary in two dimensions. Therefore, *multispecies anyons* are defined as follows: (i) an interchange of two identical particles, of species a, leads to a phase factor $\exp[\mathrm{i}\pi\alpha_{aa}]$; (ii) pulling a particle of species b around a one of species a leads to a phase factor $\exp[2\mathrm{i}\pi\alpha_{ab}]$. There is a *statistics matrix* $||\alpha_{ab}||$, which is obviously symmetric; its entries are sometimes called mutual statistics parameters.

Charge-flux composites with different values of charges and fluxes are multi-species anyons. In particular, so are quasielectrons and quasiholes in the fractional quantum Hall effect. In field theory, this is the situation with different charges and possibly several Chern-Simons gauge fields. If there is only one sort of charge and one Chern-Simons field, so we are still dealing with the theory (1)–(2), but different species a carry different charges e_a, then the statistics matrix is given by $\alpha_{ab} = e_a e_b / 2\pi k$. It is evidently symmetric, and besides its off-diagonal elements are completely determined by the diagonal ones, save for the sign: $\alpha_{ab}^2 = \alpha_{aa}\alpha_{bb}$. More generally, there may be several sorts of charges and several gauge fields; if these are numbered by β, the Lagrangian will be

$$\mathcal{L}_G = \sum_\beta \frac{1}{2} k^\beta \epsilon^{\mu\nu\lambda} A_\mu^\beta \partial_\nu A_\lambda^\beta - \sum_\beta j_\mu^\beta A^{\beta\mu} \ . \tag{4}$$

If the a-th species carries charges e_a^β, we have for the statistics matrix

$$\alpha_{ab} = \frac{\sum_\beta e_a^\beta e_b^\beta}{2\pi k^\beta} \ . \tag{5}$$

The above connection between the diagonal and off-diagonal elements no longer exists.

3 Finite-size anyons

In reality, charge-flux composites are always characterized by a finite size. For example, in the quantum Hall effect, the typical size of the flux is of the order of the magnetic length. From the point of view of field theory, we are dealing with the case [7] when the gauge field Lagrangian is not just the Chern-Simons term but rather a sum

$$\mathcal{L}_G = \mathcal{L}_0 + \frac{1}{2} k \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \ . \tag{6}$$

Most naturally, $\mathcal{L}_0$ would be the Maxwell term $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, which can either be present in the theory from the very beginning or be generated as a quantum correction [8]. There may, however, be other terms—like, for example, $(\partial_\mu A^\mu)^2$ in Higgs-like models [9].

Let there again be a pointlike charge in the origin. Solving the field equations corresponding to the Lagrangian (1), (6) yields the vector potential $A_\mu(\mathbf{r})$ produced by this charge. Assuming that $\mathcal{L}_0$ does not contain time and angle ϕ

explicitly, A_μ depends only on r. In the Lorentz gauge, consequently, A_r vanishes; A_0 is responsible for a charge-charge "Coulomb" interaction, which we will not take into account (assuming, for example, the values of the charges to be sufficiently small); now, A_ϕ, without any loss of generality, can be written

$$A_\phi(r) = -\frac{\alpha}{er}\epsilon(r) \ . \tag{7}$$

The function $\epsilon(r)$ is the form factor of the flux, because $\Phi(r) \equiv -\frac{2\pi\alpha}{e}\epsilon(r)$ is the flux through the circle of radius r. The "statistics form factor" is $\alpha\epsilon(r)$. In realistic cases, $\epsilon(r)$ will be continuous and finite everywhere; if there is no singular flux at the origin, then $\epsilon(0) = 0$, and one can normalize so that $\epsilon(\infty) = 1$. For example, in Maxwell-Chern-Simons theory [8, 10] $\epsilon(r) = 1 - krK_1(kr)$, and still $\alpha = e^2/2\pi k$.

Clearly, when the separation of anyons is much greater than their characteristic size R ($1/k$ in the example above), they are well described by the pointlike particle model; otherwise they are not. Assume that the bare particles are bosons and $\epsilon(0) = 0$. Then the quantum mechanical levels tend to those of bosons when the mean separation is much less than R and to those of anyons with statistics parameter α when it is much greater than R. The equation of state, in the high-temperature approximation, tends to that of bosons or of anyons when $\lambda \ll R$ and $\lambda \gg R$, respectively (λ is the thermal wavelength) [7, 11]. Qualitatively, this is so because in this approximation, the partition function of the Bose or Fermi gas coincides in the lowest order with that of the gas of classical particles interacting via an effective potential $v(r)$ which differs considerably from zero only for $r \lesssim \lambda$ [12]. The same holds for pointlike ideal anyons; the effective potential for them is given by $v(r) = -T \ln\left[1 + (1 - 2\alpha^2) \exp\left(-2\pi r^2/\lambda^2\right)\right]$. Therefore, the "effective statistics parameter" of finite-size anyons will be $\alpha\epsilon(r)$ averaged in some way over a range of the width of the order λ. Evidently, that tends to 0 for $\lambda \ll R$ and α for $\lambda \gg R$.

4   Anyons in a mixed Chern-Simons model

There is another Chern-Simons model, to some extent a hybrid of the ones from the previous two sections, which gives rise to anyonlike objects [13]. This is the so-called mixed Chern-Simons model, involving two species of fermions and two gauge fields, with a mixed Chern-Simons term. The Lagrangian of the model [14] is

$$\mathcal{L} = -\frac{1}{4g^2}f_{\mu\nu}f^{\mu\nu} + \bar\psi(\mathrm{i}\slashed\partial - \tau_3\slashed a - e\slashed A - \Delta)\psi - \frac{1}{4\sqrt{\partial^2}}F_{\mu\nu}F^{\mu\nu} + \frac{\mathrm{sign}(\Delta)e}{2\pi}\epsilon^{\mu\nu\rho}A_\mu f_{\nu\rho} \ . \tag{8}$$

Here

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \ , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{9}$$

A_μ is the electromagnetic field and a_μ the so-called statistical gauge field. The two species of fermions are unified in a four-component bispinor $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, and the interesting feature of the model is that its mass term $\Delta\bar{\psi}\psi = \Delta\bar{\psi}_1\psi_1 - \Delta\bar{\psi}_2\psi_2$ is P and T-invariant (see [14] for details).

The field equations corresponding to the Lagrangian (8) read

$$\frac{1}{g^2}\partial_\mu f^{\mu\nu} + \frac{se}{2\pi}\epsilon^{\nu\mu\lambda}F_{\mu\lambda} = j_3^\nu\ ,$$
$$\frac{1}{\sqrt{\partial^2}}\partial_\mu F^{\mu\nu} + \frac{se}{2\pi}\epsilon^{\nu\mu\lambda}f_{\mu\lambda} = j^\nu\ ,$$
(10)

where

$$j^\nu = \bar{\psi}\gamma^\nu\psi\ , \qquad j_3^\nu = \bar{\psi}\gamma^\nu\tau_3\psi\ , \qquad s = \text{sign}(\Delta)\ ; \tag{11}$$

Their solutions in the Lorentz gauge for pointlike sources (a purely quantum mechanical treatment is enough for our purposes)

$$j^\mu(x) = ne\delta_0^\mu\delta^2(\vec{x})\ , \qquad j_3^\mu(x) = n_3\delta_0^\mu\delta^2(\vec{x}) \tag{12}$$

are [13]

$$A_\phi(r) = -\frac{sn_3}{2er}v(fr)\ , \qquad a_\phi(r) = -\frac{sn}{2r}v(fr)\ , \tag{13}$$

where $f = (eg/\pi)^2$ and

$$u(x) = \mathbf{H}_0(x) - Y_0(x) = \frac{2}{\pi}x\,_1F_2\left(1; \frac{3}{2}, \frac{3}{2}; -\frac{x^2}{4}\right) - Y_0(x), \tag{14}$$

$$v(x) = x\int_0^\infty \exp(-x\sinh t - t)\,dt\ . \tag{15}$$

We again neglect the "Coulomb" interaction.

Now imagine a composite of n_1 fermions of the first sort (ψ_1) and n_2 of the second sort (ψ_2). According to (11) and (12), $n = n_1 + n_2$ and $n_3 = n_1 - n_2$. The phase factor arising from an interchange of two bare composites is $\exp[\text{i}\pi n^2]$, consequently they are fermions/bosons for odd/even n. However, the potentials (13) lead to a change of statistics. The "statistics form factor" as defined above is

$$\alpha\epsilon(r) = -\frac{1}{2\pi}\left[neA_\varphi(r)\cdot 2\pi r + n_3a_\varphi(r)\cdot 2\pi r\right]$$
$$= snn_3 v(fr) = s(n_1^2 - n_2^2)v(fr)\ . \tag{16}$$

The function $v(x)$ equals 0 at $x = 0$ and tends to 1 at $x \to \infty$. Therefore, at distances much greater than f^{-1} the phase factor is $\exp[\text{i}\pi((n_1 + n_2)^2 + s(n_1^2 - n_2^2))] = 1$ for any integer n_1 and n_2. Thus, at large distances the composites in question always behave as bosons. At intermediate distances, they behave like finite-size anyons. This model is interesting, as mentioned before, because of its P and T-invariance; therefore its relevance to real systems appears considerably more probable than for the usual P and T-noninvariant Chern-Simons model.

5 Conclusion

We have considered three different but to some extent interconnected generalizations of anyons, which arise from coupling charges to Chern-Simons-like gauge fields of different forms. Ideal (i.e., single-species pointlike) anyons arise as particular or limiting cases. One expects that the generalizations considered will describe the experimental situation more exactly than the ideal anyon model; it is therefore interesting to study them in more detail.

References

[1] J.M. Leinaas and J. Myrheim, Nuovo Cim. B **37** (1977) 1.

[2] G.A. Goldin, R. Menikoff, D.H. Sharp, J. Math. Phys. **21** (1980) 650; *ibid.* **22** (1981) 1664.

[3] F. Wilczek, Phys. Rev. Lett. **49** (1982) 957.

[4] F. Wilczek, Phys. Rev. Lett. **69** (1992) 132.

[5] A. Dasnières de Veigy and S. Ouvry, Phys. Lett. B **307** (1992) 91.

[6] S.B. Isakov, S. Mashkevich, and S. Ouvry, Nucl. Phys. B **448** (1995) 457.

[7] S. Mashkevich, Phys. Rev. D **48** (1993) 5953.

[8] K. Shizuya, H. Tamura, Phys. Lett. B **252** (1990) 412.

[9] X.C. Wen, A. Zee, preprint NSF-ITP-88-114 (1988).

[10] S. Mashkevich, H. Sato, G.M. Zinovjev, JETP **78** (1994) 105.

[11] S. Mashkevich, Phys. Rev. D **54** (1996) 6537.

[12] K.Huang, *Statistical Mechanics*, John Wiley & Sons, New York- London (1963).

[13] E. Gorbar, S. Mashkevich, Z. f. Phys. **C65** (1995) 705.

[14] N. Dorey, N.E. Mavromatos, Nucl. Phys. B **386** (1992) 614.